Group Theoretical Aspects of Electromagnetic Polarization and Radiative Transfer Theory

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Subject

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Assumptions on the medium in which light travels:

- Linear
- Isotropic
- Reciprocal

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Our group elements will be *polarization transformations*, usually represented by matrices.

Example of a group

Unimodular group in one dimension:

$$U(1) \triangleq \{z \in \mathbb{C} : |z| = 1\} \text{ with} \times : \text{complex multiplication.}$$

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Three observations:
(i) The elements can be written as

$$e^{i\theta}, \theta \in \mathbb{R}.$$

(ii) The resulting parameter, $\theta_1 + \theta_2$, is an analytic function of θ_1 and θ_2,

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}.$$

(iii) The set $U(1)$ describes a circle in $\mathbb{C}$,

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Such groups are called *Lie groups*.

Transversal electric field

The transversal electric field $\mathbf{E}$, of a plane harmonic electromagnetic wave with pulsation ω, wavenumber k and propagating in the positive $\mathbf{u}_z$-direction, is of the form

$$\mathbf{E} = A_x \cos\left(kz - \omega t + \varphi_x\right) \mathbf{u}_x + A_y \cos\left(kz - \omega t + \varphi_y\right) \mathbf{u}_y,$$

with amplitudes $A_x \geq 0, A_y \geq 0$ and phases $\varphi_x, \varphi_y \in \mathbb{R}$.

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It is convenient to use the complex representative ψ of the real field $\mathbf{E}$,

$$\psi = \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix},$$

with $\psi_x \triangleq A_x e^{i\varphi_x}, \psi_y \triangleq A_y e^{i\varphi_y} \in \mathbb{C}$. Complex representatives are not unique, but only determined to within a scalar phase factor.

The group SL(2,C)

Given our assumptions, a change of the transversal electric field can be represented by $\psi' = J\,\psi$ or more explicitly as

$$\begin{bmatrix} \psi'_x \\ \psi'_y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} \text{ with } a, b, c, d \in \mathbb{C},$$

with ψ', ψ Jones vectors and J a (non-singular) Jones matrix.

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Generally, $J \in GL\,(2, \mathbb{C})$, but since

$$0 = I'^2 - \left(Q'^2 + U'^2 + V'^2\right) = |\det J|^2 \left(I^2 - \left(Q^2 + U^2 + V^2\right)\right) = 0,$$

we can take $J \in SL\,(2, \mathbb{C}) \triangleq \{M \in GL\,(2, \mathbb{C}) : \det M = 1\}$.

The whole picture: breadth

All related groups

A.

Group	Pol. Space
$Spin_+(1,3)$	$\mathbb{H}^2$
$Spin_+(3,1)$	$\mathbb{R}^4$
$SV(2)$	$(Cl_{2,0})^2$
$Sp(2,\mathbb{C})$	$\mathbb{C}^2$
$SL(2,\mathbb{C})$	$\mathbb{C}^2$
$Spin(3,\mathbb{C})$	$\mathbb{C}^3$

$\simeq$: isomorph

$\xrightarrow{2\to 1}$

B.

Group	Pol. Space
$SO_+(1,3)$	$R^{1,3}$
$SO_+(3,1)$	$R^{3,1}$
$SMöb(2)$	$\mathbb{C}\cup\{\infty\}$
CS^2	S^2
$PSL(2,\mathbb{C})$	CP^1
$SO(3,\mathbb{C})$	$\mathbb{C}^3$

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Interesting $SL(2, \mathbb{C})$ representations:

- $\left(\frac{1}{2}, 0\right)$: Jones calculus
- $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$: *A new calculus...*
- $\left(\frac{1}{2}, \frac{1}{2}\right)$: Stokes vectors with (a restricted) Mueller calculus

A new polarization calculus

Based on:

- Any partially polarized light beam can be decomposed (in infinitely many ways) as the incoherent sum of two fully polarized beams.
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$$\begin{bmatrix} \psi_x'^{(1)} \\ \psi_y'^{(1)} \\ \psi_x'^{(2)} \\ \psi_y'^{(2)} \end{bmatrix} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \bar{d} & -\bar{c} \\ 0 & 0 & -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} \psi_x^{(1)} \\ \psi_y^{(1)} \\ \psi_x^{(2)} \\ \psi_y^{(2)} \end{bmatrix}.$$

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$\hookrightarrow$ *Can represent partially polarized light!*

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- Infinitely many formulations of VRT are possible!

Vectorial Radiative Transfer theory

Consequences

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Example.

Simplest VRT model: Lambert-Beer law with polarization.

Vectorial Radiative Transfer theory

Lambert-Beer law with polarization

Equation of VRT in its simplest form:

$$\frac{dS}{ds} = -ES,$$

with S : Stokes vector, E : Extinction matrix, s : distance.

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Generalized Lambert-Beer law, now including polarization.

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The mathematics tell us that:
(i) E is an element of a Lie algebra,
(ii) $\exp(-Es)$ is an element of a Lie group.

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Lambert-Beer law with polarization

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- Complication: one has to use (non-commutative) Lie algebra.
- Pitfalls: e.g., for the Lorentz group: disconnected group, non simply-connected identity group component.
- Method: use tricks from mathematics and quantum physics.

The End

THANK YOU