Mueller matrices and their role in the Vectorial Radiative Transfer model

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Talk:

- Brief overview of certain key aspects of Mueller matrices.
- Mathematical subtleties in the Vectorial Radiative Transfer model, related to the properties of Mueller matrices.

Transversal Polarization Formalisms

Jones formalism:

Light:

(i) Totally polarized waves
(ii) Represented by a complex 2×1 *Jones vector*
(iii) Formalism is linear in the electric field components.

Medium:

(i) Linear and "non-depolarizing" ($p_{out} = 1 = p_{in}$)
(ii) Represented by a complex 2×2 *Jones matrix*

Stokes/Mueller formalism:

Light:

(i) Partially polarized waves
(ii) Represented by a real 4×1 *Stokes vector*
(iii) Formalism is quadratic in the electric field components.

Medium:

(i) Linear
(ii) Represented by a real 4×4 *Mueller matrix*

Stokes Vectors

Definition

A Stokes vector $S = [I, Q, U, V]^\top$ is a real 4×1 vector satisfying:
(i) $I \geq 0$ and
(ii) $I^2 - \left(Q^2 + U^2 + V^2\right) \geq 0$.

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A convenient representation of a Stokes vector S is:

$$S = I \begin{bmatrix} 1 \\ p\mathbf{u} \end{bmatrix},$$

with *intensity* $I \geq 0$, *degree of polarization* $0 \leq p \leq 1$ and *polarization state* $\mathbf{u} \in S^2$ (Poincaré sphere).

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Stokes vectors correspond with four-momentum vectors in Special Relativity.

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Denote the set of Mueller matrices by $\mathcal{M}$.

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Characterization of $\mathcal{M}$:

- The conditions, applying to the Stokes vector components, imply that Mueller matrices will have to satisfy certain restraints.
- A numerical characterization has been given by **van der Mee**.
- A complete analytical characterization has not been given yet.

Theorem

[van der Mee, 1993] Let $M = \left[m_{ij}\right] \in M\left(4, \mathbb{R}\right)$ *satisfying* $m_{11}^2 \geq m_{12}^2 + m_{13}^2 + m_{14}^2$, $G \triangleq \mathrm{diag}\left[1, -1, -1, -1\right]$ *and* $A \triangleq GM^\top GM$. *Then* $M \in \mathcal{M}$ *iff one of the following two situations occurs:*

(i) A has one real eigenvalue λ_0, *corresponding to a time-like eigenvector, and three real eigenvalues* λ_1, λ_2, λ_3, *corresponding to space-like eigenvectors, and* $\lambda_0 \geq \max(0, \lambda_1, \lambda_2, \lambda_3)$.

(ii) A has four real eigenvalues λ, λ, μ *and* ν *but is not diagonalizable. The eigenvectors corresponding to* μ *and* ν *are space-like and to the double eigenvalue* λ *corresponds a Jordan block of size* 2 *with positive sign. Moreover,* $\lambda \geq \max(0, \mu, \nu)$.

- The structure $(\mathcal{M}, +, \times)$ is a semi-ring over $\mathbb{R}_{\geq 0}$.

Mueller Matrices

Some mathematical properties

- The structure $(\mathcal{M}, +, \times)$ is a semi-ring over $\mathbb{R}_{\geq 0}$.
- In particular, invertible Mueller matrices form a *group*, denoted $\mathcal{M}_G$, and which is a *Lie* group. A Lie group is: (i) algebraically a *group* and (ii) geometrically a (smooth) *manifold*.

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- We know that $SO_+(1,3) \subset O_+(1,3) \subset \mathcal{M}_G \subset \mathcal{M}$. Notation: $SO_+(1,3)$ = the *proper orthochronous Lorentz group*, $O_+(1,3)$ = the *orthochronous Lorentz group* (and $O(1,3)$ = the *Lorentz group*).

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- Elements in $\mathcal{M}_G \backslash O_+ (1,3)$ contain 16 real parameters, while elements in $O_+ (1,3)$ contain 6 real parameters. Hence, the full group $\mathcal{M}_G$ is much larger than $O_+ (1,3)$.

Vectorial Radiative Transfer (VRT)

Equation

$$
\begin{aligned}
(\mathbf{u} \cdot \nabla) S(\mathbf{x};\mathbf{u}) &= -K(\mathbf{x}) S(\mathbf{x};\mathbf{u}) \\
&\quad + \int_{S^2} Z\left(\mathbf{x};\mathbf{u},\mathbf{u}'\right) S\left(\mathbf{x};\mathbf{u}'\right)\, d\mathbf{u}' + B(\mathbf{x};\mathbf{u}),
\end{aligned}
$$

$\mathbf{x}$: position vector,
$\mathbf{u}$: unit vector defining the line of sight,
S: Stokes vector field (the unknown), [4x1],
K: *extinction matrix*, [4x4],
Z: *phase matrix*, [4x4] ($\in \mathcal{M}$),
B: emission vector field, [4x1],

+ boundary conditions.

Vectorial Radiative Transfer theory

- Combining the transversal polarization of partially coherent plane waves, in terms of the Stokes/Mueller formalism, with the phenomenological theory of (stationary) scalar radiative transfer goes back to [Chandrasekhar 1950]. The result is the well-known Vectorial Radiative Transfer (VRT) equation.

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- These subtleties are related to the global topology of the manifold of the underlying Lie group of Mueller matrices.
- As a consequence, a disagreement can arise for certain media between:
  (i) the solution of the VRT equation and
  (ii) the in situ measurement of the Stokes vector.

Vectorial Lambert-Beer (VLB) law

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A. Infinitesimal model

In this medium and with absorption represented by the extinction matrix K, the Stokes vector is believed to satisfy the equation

$$\frac{d}{dz} S(z) = -K(z) S(z) . \tag{1}$$

Eq. (1) describes the transport of S along z through our medium over an infinitesimal distance dz. Eq. (1) is the *infinitesimal model*.

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B. Finite model
From an experimental point of view, one can always measure the Mueller matrix $M(z_{out}, z_{in})$ which relates the Stokes vectors at z_{out} and z_{in},

$$S(z_{out}) = M(z_{out}, z_{in})\, S(z_{in})\,. \tag{2}$$

Eq. (2) describes the transport of S along z through our medium over a finite distance $z_{out} - z_{in}$. Eq. (2) is the *finite model*.

A Fundamental Question

Question: **Is the infinitesimal model (eq. (1)) always equivalent to the finite model (eq. (2))** ?

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Answer: **No**.

How Can It Go Wrong?

Assuming a constant extinction matrix K, the solution of the infinitesimal model is

$$S\left(z_{out}\right) = \exp\left(-K\ \left(z_{out} - z_{in}\right)\right) S\left(z_{in}\right).$$

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So, if our medium is one that is characterized by an unreachable Mueller matrix, then any solution method (either numerically or analytically) will produce the wrong answer.

$\Rightarrow$ disagreement at experimental validation!

Example

Phase conjugation medium

Consider a medium, acting on the Jones vector J of a traversing plane wave, by the operator C, (overbar = complex conjugation), with

$$C : J_{in} \mapsto J_{out} = \bar{J}_{in}.$$

This is a linear operation (over $\mathbb{R}$), but one that cannot be represented by a Jones matrix.

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Applying Γ, we obtain from C an operator D, acting on Stokes vectors, which turns out to be

$$D : S_{in} \mapsto S_{out} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} S_{in}.$$

The appearing Mueller (transmission!) matrix is clearly outside the range of the the exponential function (it's an element of $O_+ (1, 3)$).

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Phase conjugation medium – cont'd

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Such a medium can be made by letting two laser beams interact in certain non-linear optical media [1]. The effect of this setup on a third (low power) signal wave, propagating through this medium, is the creation of a wave with reversed phase lag (i.e., a phase conjugation). W.r.t. the signal wave, the medium acts approximative linear.

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Occurence:

- An optical component, inserted between long optical fibers by telecom engineers to correct phase front distortions caused by the fibers, contains a phase conjugation medium.
- Phase conjugation has been observed in several metal vapors (refs 93–99 in [1]). Extra care may therefore be necessary when doing VRT calculations in the atmosphere of certain hot exo-planets.

[1] G.S. He, Optical phase conjugation principles techniques and applications, *Progress in Quantum Electronics, 26, 131–191, 2002.*

A First Conclusion

A phase conjugate medium is a physical medium:

- in which the VRT equation fails and
- which shows that the general form of the VLB law cannot be just a straight forward generalization of its scalar counterpart.

Vectorial Lambert-Beer (VLB) law

General form

Insight in this problem is provided by considering the topology of the manifold of the Lie group of Mueller matrices.

This suggests that the VLB law is generally of the form

$$S\left(z_{out}\right) = M_0 \exp\left(-K\ \left(z_{out} - z_{in}\right)\right) S\left(z_{in}\right),$$

wherein M_0 corresponds to the point on the Mueller matrix manifold, at which is tangent the plane, on which we formulate our infinitesimal model.
The value of the extinction matrix K, used in the infinitesimal model, generally will depend on the choice of M_0.

M_0 must be chosen close enough to the Mueller matrix that relates $S\left(z_{in}\right)$ to $S\left(z_{out}\right)$ (i.e., “in the right neighborhood”).

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- Is a consequence of the global *topology* of the manifold.
- Key concepts: (i) *connectedness*, (ii) *compactness* and (iii) *simply connectedness* of the manifold.

Lie Group Topology Effects

- If a Lie group manifold is *not connected*, then *exp* cannot reach group elements on other components.

Example: full Lorentz group $O\left(1,3\right)$: 4 components. Its subgroup $O_{+}\left(1,3\right)$: 2 components.

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- The map *exp* may exceptionally be surjective.

Example: the identity component $SO_+(1,3)$ of the Lorentz group.

Summary of the VRT problem

The illness:

- The VRT equation, being an infinitesimal model, is a local model.
- If the Lie group underlying an equation has trivial topology, then: infinitesimal model $\Leftrightarrow$ finite model.
- The group of Mueller matrices underlying the VRT problem is not fully known, but it is already known that it has non-trivial topology (not connected, non-compact, not simply connected).
- Being not connected is already sufficient that infinitesimal VRT model $\not\Leftrightarrow$ finite model!

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The cure:

- Supply the information that got stripped away when formulating the infinitesimal model.
- The lost information is: the global structure of the manifold of Mueller matrices.
- Determine where on the manifold the Mueller matrices of the medium are located (i.e., choose the right "neighborhood").
- Reformulate the VRT equation on the tangent plane at an element of this neighborhood and solve as usual!

Final Conclusion

Question: **How far can we trust the VRT equation?**
Answer: **Only as far as the local neighborhood.**

Since the complete topology of the full group of Mueller matrices is not yet fully known, we cannot specify in general how far a neighborhood extents.

The End

THANK YOU

TTP and STR Correspondence

Quantity	TTP	STR
I	Intensity	Rel. energy E divided by c
$I_p \triangleq pI$	Polarization intensity	Rel. momentum $\|\mathbf{p}\|$
p	Degree of polarization	Normalized speed $\|\mathbf{v}\| / c$
$\mathbf{u}$	Polarization state	Unit velocity vector $\mathbf{v} / \|\mathbf{v}\|$
$\beta \triangleq \operatorname{artanh} p$	Lorentzian angle of pol.	Rapidity β
$\gamma \triangleq \frac{1}{\sqrt{1-p^2}}$	Lorentzian factor of pol.	Time dilatation factor γ
$\|S\|_{1,3}$	Lorentzian length of S	Rest energy E_0 divided by c

Table : Correspondence between a **Stokes vector** $S = I\left[1, p\mathbf{u}\right]^\top$ in the Theory of Transversal Polarization (TTP) and the **four-momentum vector** $P = \left[E/c, \mathbf{p}\right]^\top$ in the Special Theory of Relativity (STR), of a *uniformly moving particle* with rest mass $m_0 = I/\left(\gamma c\right)$, relativistic mass $m = \gamma m_0 = I/c$, velocity vector $\mathbf{v} = \left(pc\right)\mathbf{u}$ and relativistic momentum vector $\mathbf{p} = m\mathbf{v} = I_p\mathbf{u}$.

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- Alas, in the STR literature, the full causality group has not yet been identified!
- Both in STR and in the TTP, it has been established that the sought group contains $O_+(1,3)$ as a subgroup, but neither community has found so far the full extent of this common group!

Mueller Matrices

Which derives from a Jones matrix

A Mueller matrix which derives from a Jones matrix is called a *Jones-Mueller matrix*. Invertible Jones-Mueller matrices have the form $M = aL$, with $a > 0$ and $L \in SO_+(1,3)$. They form the subgroup $\mathcal{M}_G^J = \mathbb{R}_{>0}^\times \times SO_+(1,3) \subset \mathcal{M}_G$, which elements contain 7 real parameters.

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Explicitly, any $M \in \mathcal{M}_G^J$ is of the form

$$M = a\gamma \begin{bmatrix} 1 & p\mathbf{x}^\top \\ p\mathbf{y} & \gamma^{-1}R + \left(1 - \gamma^{-1}\right)\mathbf{y}\mathbf{x}^\top \end{bmatrix},$$

with $\mathbf{x}, \mathbf{y}$ Euclidean unit vectors, $0 \leq p < 1$, $1 \leq \gamma \triangleq 1/\sqrt{1-p^2} < +\infty$, $R \in SO(3)$ and $\mathbf{y} = R\mathbf{x}$.

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- The subgroup corresponding with $a = 1$ and $p = 0$ is $SO(3)$ and represents *retarders* (birefringence).
- The subset corresponding with $a = 1$ and $R = I_3$ are Lorentz boost matrices, which represent *diattenuators* (dichroism).

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- and $\forall g \in S$ exists an *inverse* element $g^{-1} \in S$ such that $g \times g^{-1} = 1 = g^{-1} \times g$.