The diffusion approximation for Rayleigh scattering

Hélène Frisch

Laboratoire Joseph Louis Lagrange, Observatoire de la Côte d'Azur

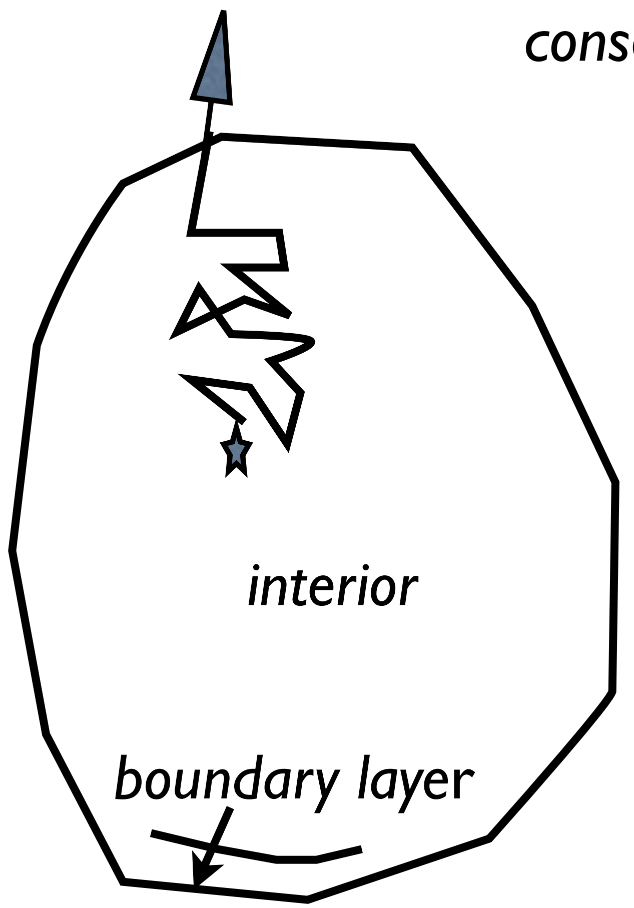
Asymptotic expansion of the radiative transfer equation

➤ approximate solutions for monochromatic scattering

valid if photons under go a large number of scatterings during their life-time

i.e. mean free-path of photons << dimension of the medium

conservative or nonconservative media , if the destruction probability ϵ *is small*



(1) Interior problem

radiation field can be derived from a diffusion equation

at leading order it is isotropic and unpolarized

(2) Boundary layer problem

semi-infinite , conservative, 1D medium

matching the interior and boundary layer ⟶

boundary conditions for the diffusion equation and the boundary layer problem

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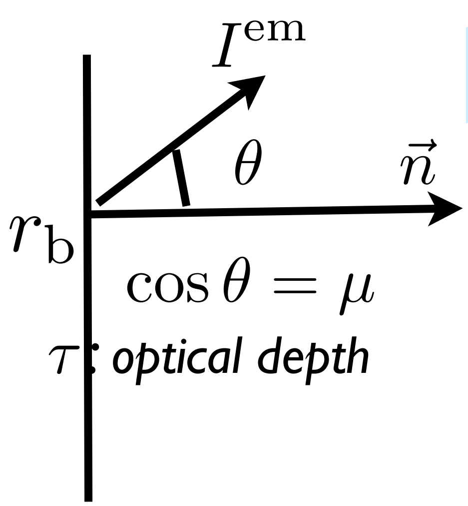
Exact and asymptotic results (monochromatic)

Rayleigh scattering

polarized radiation field : $\vec{I} = (I_{\rm l}, I_{\rm r})$ *Chandrasekhar notation : l parallel, r perpendicular*



Exact results semi-infinite 1D medium, cylindrical symmetry, conservative $(\epsilon = 0)$

- "Milne problem" $I_{\rm l}^{\rm em}(\mu) = \frac{3F}{8\pi}\frac{q}{\sqrt{2}}H_{\rm l}(\mu)$ $I_{\rm r}^{\rm em}(\mu) = \frac{3F}{8\pi}\frac{c+\mu}{\sqrt{2}}H_{\rm r}(\mu)$

$H_{\rm l}(\mu), H_{\rm r}(\mu)$ *H - functions for Rayleigh scattering;* q, c *constants;* $F = F_{\rm l} + F_{\rm r}$

- "Diffuse reflection" Incident radiation : $I^{\rm inc}(\mu) = (I_{\rm l}^{\rm inc}(\mu), I_{\rm r}^{\rm inc}(\mu)), \quad \mu < 0$

at infinity $I(\infty) = \frac{\sqrt{3}}{4}\int_0^1 [qH_{\rm l}(\mu)I_{\rm l}^{\rm inc}(-\mu) + (\mu + c)H_{\rm r}(\mu)I_{\rm r}^{\rm inc}(-\mu)]\mu\, d\mu$ isotropic, unpolarized

emergent radiation

$$I_{\rm l}^{\rm em}(\mu) = H_{\rm l}(\mu)\int_0^1 [f_{\rm l}^{\rm l}(\mu,\mu')H_{\rm l}(\mu')I_{\rm l}^{\rm inc}(-\mu') + f_{\rm l}^{\rm r}(\mu,\mu')H_{\rm r}(\mu')I_{\rm r}^{\rm inc}(-\mu')]\, d\mu'$$

$$I_{\rm r}^{\rm em}(\mu) = H_{\rm r}(\mu)\int_0^1 [f_{\rm r}^{\rm l}(\mu,\mu')H_{\rm l}(\mu')I_{\rm l}^{\rm inc}(-\mu') + f_{\rm r}^{\rm r}(\mu,\mu')H_{\rm r}(\mu')I_{\rm r}^{\rm inc}(-\mu')]\, d\mu'$$

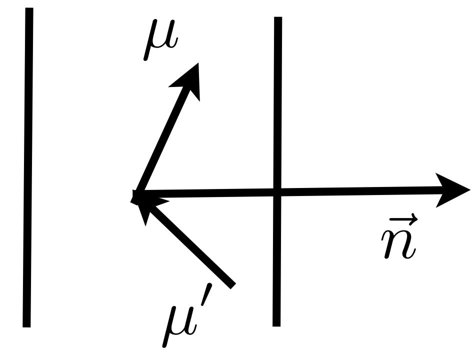
Asymptotic result finite 1D medium with $\epsilon = 0$ or $\epsilon \neq 0$,

no incident radiation
internal source

$$I_{\rm l}^{\rm em}(\mu) \simeq \frac{q}{\sqrt{2}}H_{\rm l}(\mu)\frac{dI^{\rm int}}{d\tau}|_{r_{\rm b}} \qquad I_{\rm r}^{\rm em}(\mu) \simeq \frac{\mu + c}{\sqrt{2}}H_{\rm r}(\mu)\frac{dI^{\rm int}}{d\tau}|_{r_{\rm b}}$$

Radiative transfer equation for Rayleigh scattering

1D slab



T < τ < 0

Radiative transfer equation

nonconservative : $\epsilon \neq 0$

$$\mu \frac{\partial \vec{I}(\tau,\mu)}{\partial \tau} = (1-\epsilon)\mathcal{L}[\vec{I}(\tau,\mu)] + \epsilon[\vec{I}(\tau,\mu) - \vec{Q}(\tau)].$$

$$\mathcal{L}[\vec{I}(\tau,\mu)] \equiv \vec{I}(\tau,\mu) - \frac{1}{2}\int_{-1}^{+1} R(\mu,\mu')\vec{I}(\tau,\mu')\,d\mu'$$ *conservative scattering operator*

$$R(\mu,\mu') = \frac{3}{4}\begin{bmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{bmatrix}$$ *Rayleigh phase matrix*

μ', μ *: direction of incident and scattered beams*

Properties of $\mathcal{L}$

- kernel $\mathcal{L}[\vec{f}(\mu)] = 0$ isotropic and unpolarized vectors : $f_{\mathrm{l}}(\mu) = f_{\mathrm{r}}(\mu) = f$

- solutions of $\mathcal{L}[\vec{I}(\tau,\mu)] = \vec{V}(\tau,\mu)$ *solvability condition ?*

$$\int_{-1}^{+1} \mathcal{L}[\vec{f}(\mu)]\,d\mu = \bar{\mathcal{L}} = [\bar{\mathcal{L}}_{\mathrm{l}}, \bar{\mathcal{L}}_{\mathrm{r}}]$$ *the components satisfy* $\bar{\mathcal{L}}_{\mathrm{l}} + \bar{\mathcal{L}}_{\mathrm{r}} = 0$ $\longleftrightarrow$ radiative flux constant

$$\int_{-1}^{+1} \vec{V}(\tau,\mu)\,d\mu = \vec{v}(\tau) = [v_{\mathrm{l}}(\tau), v_{\mathrm{r}}(\tau)]$$ solvability condition : $v_{\mathrm{l}}(\tau) + v_{\mathrm{r}}(\tau) = 0$

Interior solution

Interior solution : low resolution description of radiation field

scale of variation $\sim$ thermalization length $\sim 1/\sqrt{\epsilon}$

- rescaled optical depth $\tilde{\tau} = \eta\tau$ with $\eta = \sqrt{\epsilon}$ η is the expansion parameter

  $\tilde{\tau}$ *is order of unity for* τ *order of the thermalization length*

- rescaled radiation field and asymptotic expansion

$$\vec{I}(\tau,\mu) = \vec{I}(\tilde{\tau},\mu) = \vec{I}_0(\tilde{\tau},\mu) + \eta\vec{I}_1(\tilde{\tau},\mu) + \eta^2\vec{I}_2(\tilde{\tau},\mu) + \ldots$$

- rescaled radiative transfer equation

$$\eta\mu\frac{\partial\vec{I}(\tilde{\tau},\mu)}{\partial\tilde{\tau}} = (1-\eta^2)\mathcal{L}[\vec{I}(\tilde{\tau},\mu)] + \eta^2[\vec{I}(\tilde{\tau},\mu) - \vec{Q}(\tilde{\tau})].$$

regrouping terms with the same order of η → *hierarchy of equations*

η^0 $\mathcal{L}[\vec{I}_0] = 0$ → $\vec{I}_0(\tilde{\tau},\mu) = \vec{I}_0(\tilde{\tau}) = (i_0(\tilde{\tau}), i_0(\tilde{\tau}))$ the leading term is isotropic and unpolarized

η^1 → $\vec{I}_1(\tilde{\tau},\mu) = \mu\,\partial\vec{I}_0(\tilde{\tau})/\partial\tilde{\tau} + \vec{c}_1(\tilde{\tau})$ *with* $\vec{c}_1(\tilde{\tau})$ *undetermined, in the kernel of* $\mathcal{L}$

η^2 solvability condition → $\frac{1}{3}\frac{d^2 i_0(\tilde{\tau})}{d\tilde{\tau}^2} + q(\tilde{\tau}) - i_0(\tilde{\tau}) = 0$ diffusion equation

for an unpolarized source term $\vec{Q}(\tilde{\tau}) = (q(\tilde{\tau}), q(\tilde{\tau}))$

At leading order the radiation field is unpolarized; the first order correction creates the polarization

The boundary layer analysis → *the boundary conditions for the diffusion equation*

The boundary layer and asymptotic matching (I)

The optical thickness of the boundary layer is of order η in the interior variable $\tilde{\tau}$

stretching in the direction perpendicular to the boundary

the boundary layer variable : $s = \tilde{\tau}/\eta, \quad s \in [0, +\infty[$

- The radiation field in the boundary layer : $\vec{I}(\tau, \mu) = \vec{I}^{\rm int}(\tilde{\tau}, \mu) + \vec{I}^{\rm b}(s, \mu)$
- Coupling with the interior solution :

condition at infinity : $\vec{I}^{\rm b}(s, \mu) \to 0, \quad s \to \infty$

boundary condition for $\vec{I}(\tau, \mu)$ *is applied to the sum* $\vec{I}^{\rm int} + \vec{I}^{\rm b}$

- Asymptotic expansion : $\vec{I}^{\rm b}(s, \mu) = \vec{I}^{\rm b}_0(s, \mu) + \eta \vec{I}^{\rm b}_1(s, \mu) + \dots$
- Radiative transfer equation $\mu \dfrac{\partial \vec{I}^b_k(s, \mu)}{\partial s} = \mathcal{L}[\vec{I}^b_k(s, \mu)], \quad k = 0, 1$

boundary condition $\vec{I}^{\rm b}_k(0, \mu) = -\vec{I}^{\rm int}_k(0, \mu), \quad \mu < 0, \quad k = 0, 1$

if the incident radiation is zero on the boundaries of the medium

Diffuse reflection problems in a 1D semi-infinite conservative medium

The boundary layer and asymptotic matching (II)

η^0 condition at infinity $\longrightarrow$ $\vec{I}_0^{\rm int}(0) = 0$

(i) boundary condition for the interior diffusion equation $\vec{I}_0^{\rm int}(0) = 0$

(ii) boundary layer solution is zero at leading order $I_0^{\rm b}(s,\mu) = 0$

η^1 condition at infinity $\longrightarrow$ $\vec{I}_1^{\rm int}(0,\mu) = (\mu + L)\partial \vec{I}_0(\tilde{\tau})/\partial\tilde{\tau}|_{\tilde{\tau}=0}$ *with* L *a constant*

solution of the diffuse reflection problem with $\vec{I}_1^{\rm inc}(0,\mu) = -I_1^{\rm int}(0,\mu)$ $\longrightarrow$

emergent intensity

$$I_{\rm l}^{\rm em}(\mu) \simeq \sqrt{\epsilon}\frac{q}{\sqrt{2}} H_{\rm l}(\mu)\frac{di_0^{\rm int}(\tilde{\tau})}{d\tilde{\tau}}|_{\tilde{\tau}=0} \qquad I_{\rm r}^{\rm em}(\mu) \simeq \sqrt{\epsilon}\frac{\mu + c}{\sqrt{2}} H_{\rm r}(\mu)\frac{di_0^{\rm int}(\tilde{\tau})}{d\tilde{\tau}}|_{\tilde{\tau}=0}$$

$c = 0.873, q = 0.690, L = 0.71$

Improved interior solution

Direction averaged field $\vec{J}(\tilde{\tau}) = \frac{1}{2}\int_{-1}^{+1} [\vec{I}_0(\tilde{\tau}) + \eta \vec{I}_1(\tilde{\tau},\mu)]\, d\mu$ *satisfies the diffusion equation*

with the mixed type boundary condition $\vec{J}(0) - L\frac{d\vec{J}(\tau)}{d\tau}|_{\tau=0} = 0$

Concluding remarks

- **Rii partial frequency redistribution :** asymptotic analysis at large frequencies

Frisch & Bardos 1981
scalar case , 3-dimension
no incident radiation

➔ interior : *space and frequency diffusion equation*

➔ boundary layer : *diffuse reflection for monochromatic conservative scattering*

➔ emergent intensity $I^{\rm em}(0,\mu,x) \simeq \frac{H(\mu)}{\sqrt{3}} \frac{\partial I^{\rm int}}{\partial \tau_x}|_{\tau_x=0}$

τ_x : *monochromatic optical depth*
$H(\mu)$: *Chandrasekhar function for conservative scattering*

- **Rii with Rayleigh scattering 1D**

$$I_{\rm l}^{\rm em}(\mu,x) \simeq \frac{q}{\sqrt{2}} H_{\rm l}(\mu) \frac{\partial I^{\rm int}}{\partial \tau_x}|_{\tau_x=0} \qquad I_{\rm r}^{\rm em}(\mu,x) \simeq \frac{\mu+c}{\sqrt{2}} H_{\rm r}(\mu) \frac{\partial I^{\rm int}}{\partial \tau_x}|_{\tau_x=0}$$

results valid at frequencies $\sim a^{1/3}\epsilon^{-1/3}$ *, around 4-5 or more Doppler widths*

a : *Voigt parameter*

- **3D Monochromatic and Rii Rayleigh scattering**

  - no incident radiation : *same results as in the 1D case, with a 3D interior diffusion equation*
  - incident radiation $\vec{I}^{\rm ext}$

    ➔ *the emergent intensity is solution of a diffuse reflection problem with an incident radiation* $\vec{I}^{\rm ext}$

    ➔ *the boundary condition for the interior diffusion equation is the value at infinity of the solution of the diffuse reflection problem*

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